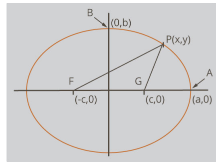


Figure 2.3.1-2

Given: Two fixed points F, G and *a fixed positive number* k. The ellipse consists of all points P such that $\overline{FP} + \overline{GP} = k$.

The fixed points F and G have coordinates $(-c, 0)$ and $(c, 0)$, respectively. The points A and B are points where the ellipse intersects the positive x-axis and positive y-axis, respectively.

1. Use the distance formula to express the relationship $\overline{FP} + \overline{GP} = k$ in terms of the coordinates of an arbitrary point $P(x, y)$, as pictured, which lies on the ellipse. (Your expression should involve a sum of two square roots.)

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\overline{FP} = \sqrt{(x+c)^2 + (y-0)^2}$$

$$\boxed{\overline{FP} = \sqrt{(x+c)^2 + y^2}}$$

$$\boxed{\overline{GP} = \sqrt{(x-c)^2 + y^2}}$$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = K$$

2. Use algebraic techniques to eliminate the square roots that occur in 1. (Note: This involves squaring twice; it helps to simplify the result obtained by squaring the first time before squaring a second time.) Your expression should involve x, y, c, and k.

$$\sqrt{(x+c)^2 + y^2} = K - \sqrt{(x-c)^2 + y^2}$$

$$\left(\sqrt{(x+c)^2 + y^2}\right)^2 = \left(K - \sqrt{(x-c)^2 + y^2}\right)^2$$

$$(x+c)^2 + y^2 = \left(K - \sqrt{(x-c)^2 + y^2}\right)\left(K - \sqrt{(x-c)^2 + y^2}\right)$$

$$(x+c)^2 + \cancel{y^2} = K^2 - 2K\sqrt{(x-c)^2 + y^2} + (x-c)^2 + \cancel{y^2}$$

$$(x+c)(x+c)$$

$$\cancel{x^2} + 2xc + \cancel{c^2} = K^2 - 2K\sqrt{(x-c)^2 + y^2} + \cancel{x^2} - \cancel{2xc} + \cancel{c^2}$$

$$+2xc \qquad\qquad +\cancel{2xc}$$

$$4xc = \cancel{K^2} - 2K\sqrt{(x-c)^2 + y^2}$$

$$-K^2 \qquad -\cancel{K^2}$$

$$4xc - K^2 = -2K\sqrt{(x-c)^2 + y^2}$$

$$(4xc - K^2)^2 = \left(-2K\sqrt{(x-c)^2 + y^2}\right)^2$$

$$(4xc - K^2)(4xc - K^2)$$

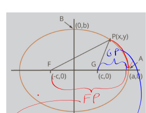
$$16x^2c^2 - 8K^2xc + K^4 = 4K^2\left((x-c)^2 + y^2\right)$$

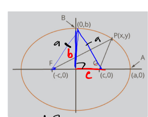
$$16x^2c^2 - 8K^2xc + K^4 = 4K^2(x^2 - 2xc + c^2 + y^2)$$

$$16x^2c^2 - \cancel{8K^2xc} + K^4 = 4K^2x^2 - \cancel{8K^2xc} + 4K^2c^2 + 4K^2y^2$$

$$16x^2c^2 + K^4 = 4K^2x^2 + 4K^2c^2 + 4K^2y^2$$

3. **Verify** that $k = 2a$ and $c^2 = a^2 - b^2$, knowing that A and B are points on the ellipse.





$$\overline{FP} = a + \cancel{c}$$

$$\overline{GP} = a - \cancel{c}$$

$$K = 2a$$

$$\cancel{b^2} + c^2 = a^2 - b^2$$

$$-\cancel{b^2}$$

4. Substitute the values for k and c into your derived equation to obtain the standard equation of the ellipse centered at the origin with semi-major axis of length a and semi-minor axis of length b:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$16x^2c^2 + K^4 = 4K^2x^2 + 4K^2c^2 + 4K^2y^2$$

$$16x^2(a^2 - b^2) + (2a)^4 = 4(2a)^2x^2 + 4(2a)^2(a^2 - b^2) + 4(2a)^2y^2$$

$$\cancel{16x^2a^2} - 16x^2b^2 + 16a^4 = \cancel{16a^2x^2} + 16a^2(a^2 - b^2) + 16a^2y^2$$

$$-16x^2b^2 + \cancel{16a^4} = \cancel{16a^4} - 16a^2b^2 + 16a^2y^2$$

$$\frac{-16x^2b^2 = -16a^2b^2 + 16a^2y^2}{-16}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$\frac{x^2\cancel{b^2}}{a^2\cancel{b^2}} = \frac{\cancel{a^2}b^2 - \cancel{a^2}y^2}{\cancel{a^2}b^2}$$

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$