HW 2.2.0: Arithmetic of Complex Numbers

We will begin with a review of the definition of complex numbers.

Imaginary Number i

The most basic complex number is i, defined to be $i=\sqrt{-1}$, commonly called an **imaginary number**. Any real multiple of i is also an imaginary number.

Example 1

Simplify $\sqrt{-9}$.

We can separate $\sqrt{-9}$ as $\sqrt{9}\sqrt{-1}$. We can take the square root of 9, and write the square root of -1 as i.

$\sqrt{-9}=\sqrt{9}\sqrt{-1}=3i$

A complex number is the sum of a real number and an imaginary number.

Complex Number

A **complex number** is a number $z=a+bi$, where a and b are real numbers

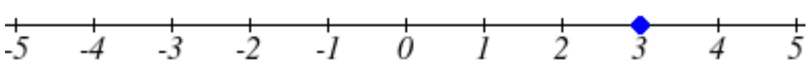
a is the real part of the complex number

b is the imaginary part of the complex number

$i=\sqrt{-1}$

Plotting a complex number

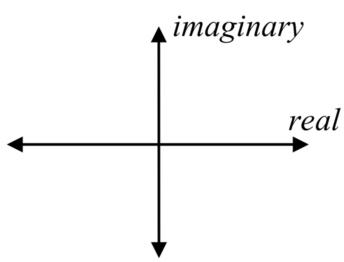
We can plot real numbers on a number line. For example, if we wanted to show the number 3, we plot a point:



To plot a complex number like $3-4i$, we need more than just a number line since there are two components to the number. To plot this number, we need two number lines, crossed to form a complex plane.

Complex Plane

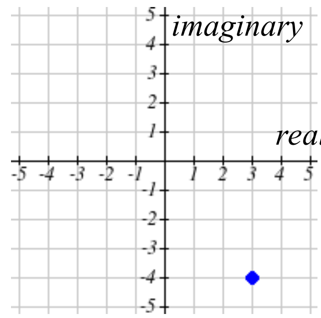
In the **complex plane**, the horizontal axis is the real axis and the vertical axis is the imaginary axis.



Example 2

Plot the number $3-4i$ on the complex plane.

The real part of this number is 3, and the imaginary part is -4. To plot this, we draw a point 3 units to the right of the origin in the horizontal direction and 4 units down in the vertical direction.



Because this is analogous to the Cartesian coordinate system for plotting points, we can think about plotting our complex number $z=a+bi$ as if we were plotting the point (a, b) in Cartesian coordinates. Sometimes people write complex numbers as $z=x+yi$ to highlight this relation.

Arithmetic on Complex Numbers

Before we dive into the more complicated uses of complex numbers, let's make sure we remember the basic arithmetic involved. To add or subtract complex numbers, we simply add the like terms, combining the real parts and combining the imaginary parts.

Example 3

Add $3-4i$ and $2+5i$.

Adding $(3-4i)+(2+5i)$, we add the real parts and the imaginary parts

$3+2-4i+5i$

$5+i$

Try it Now

1. Subtract $2+5i$ from $3-4i$.

We can also multiply and divide complex numbers.

Example 4

Multiply: $4(2+5i)$.

To multiply the complex number by a real number, we simply distribute as we would when multiplying polynomials.

$$4(2+5i)$$
$$=4\cdot 2+4\cdot 5i$$
$$=8+20i$$

Example 5

Divide $\dfrac{(2+5i)}{(4-i)}$.

To divide two complex numbers, we have to devise a way to write this as a complex number with a real part and an imaginary part.

We start this process by eliminating the complex number in the denominator. To do this, we multiply the numerator and denominator by a special complex number so that the result in the denominator is a real number. The number we need to multiply by is called the **complex conjugate**, in which the sign of the imaginary part is changed. Here, $4+i$ is the complex conjugate of $4-i$. Of course, obeying our algebraic rules, we must multiply by $4+i$ on both the top and bottom.

$$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}$$

To multiply two complex numbers, we expand the product as we would with polynomials (the process commonly called FOIL – “first outer inner last”). In the numerator:

$(2+5i)(4+i)$	Expand
$=8+20i+2i+5i^2$	Since $i=\sqrt{-1}$, $i^2=-1$
$=8+20i+2i+5(-1)$	Simplify
$=3+22i$	

Following the same process to multiply the denominator

$(4-i)(4+i)$	Expand
$(16-4i+4i-i^2)$	Since $i=\sqrt{-1}$, $i^2=-1$
$(16-(-1))$	
$=17$	

Combining this we get $\dfrac{3+22i}{17}=\dfrac{3}{17}+\dfrac{22i}{17}$

Try it Now

2. Multiply $3-4i$ and $2+3i$.

HW Practice:
Simplify each expression to a single complex number.

1. $\sqrt{-16}$

2. $\sqrt{-25}$

3. $\sqrt{-8}\sqrt{-18}$

4. $\sqrt{-7}\sqrt{-28}$

5. $\dfrac{3+\sqrt{-24}}{3}$

6. $\dfrac{6+\sqrt{-45}}{3}$

Simplify each expression to a single complex number.

7. $(4+3i)+(6-4i)$

8. $(-1-3i)+(2+7i)$

9. $(-4+4i)-(7-2i)$

10. $(1-4i)-(2+i)$

11. $(6+7i)(3i)$

12. $(8-4i)(6i)$

13. $(8-9i)(7)$

14. $(-4+7i)(10)$

15. $(12+9i)(5-3i)$

16. $(-6+12i)(-1+7i)$

17. $(3-5i)(3+5i)$

18. $(6+7i)(6-7i)$

19. $\dfrac{9+7i}{4}$

20. $\dfrac{1-4i}{8}$

21. $\dfrac{-6+2i}{i}$

22. $\dfrac{9+11i}{2i}$

23. $\dfrac{13-2i}{9+2i}$

24. $\dfrac{10+3i}{3-2i}$

25. i^4

26. i^{15}

27. i^{21}

28. i^{30}