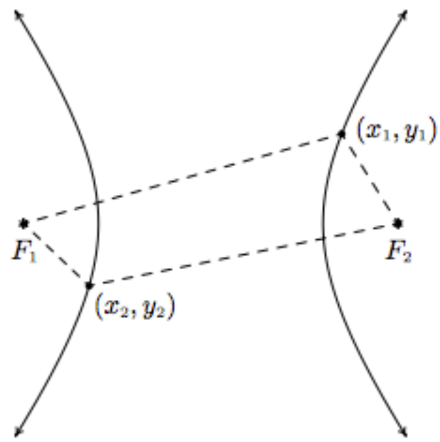
2.3 Conic Sections: Hyperbola

Hyperbola (locus definition) Set of all points (x, y) in the plane such that the absolute value of the difference of each distances from F_1 and F_2 to (x, y) is a constant distance, *d*.



In the figure above:

The distance from F_1 to (x_1, y_1) - the distance from F_2 to $(x_1, y_1) = d$

and

The distance from F_1 to (x_2, y_2) - the distance from F_2 to $(x_2, y_2) = d$

Standard Form of a Hyperbola:

Horizontal Hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Vertical Hyperbola

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

center = (h, k)
2a = distance between vertices
c = distance from center to focus
$c^2 = a^2 + b^2$
eccentricity $e = \frac{c}{a}$ ($e > 1$ for a hyperbola)
Conjugate axis = *2b*
Transverse axis = *2a*

Horizontal Asymptotes

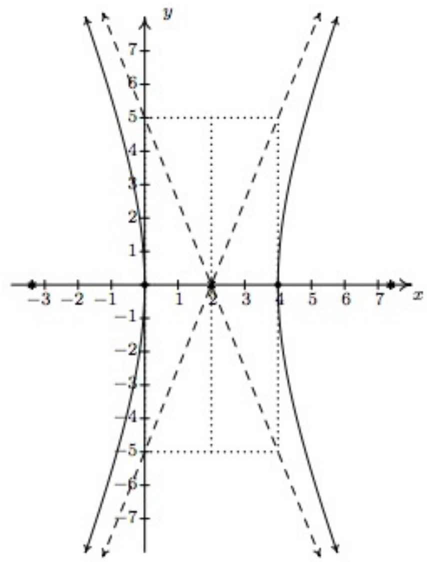
$$y - k = \pm\frac{b}{a}(x-h)$$

Vertical Asymptotes

$$y - k = \pm\frac{a}{b}(x-h)$$

Show how $d = 2a$

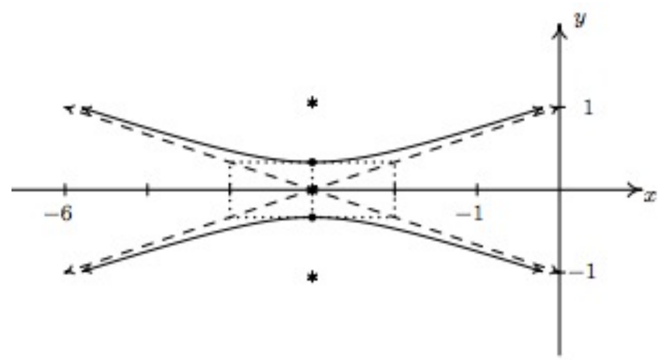
Ex. Graph $\frac{(x-2)^2}{4} - \frac{y^2}{25} = 1$



Center: (2, 0)
Vertices (4, 0) & (0, 0)
Foci $\left(2 \pm \sqrt{29}, 0\right)$

Asymptotes: $y = \pm\frac{5}{2}(x-2)$

Ex. Graph $9y^2 - x^2 - 6x - 10 = 0$



$$9y^2 - x^2 - 6x = 10$$
$$9y^2 - 1\left(x^2 + 6x + \underline{9}\right) = 10 - \underline{9}$$
$$9y^2 - 1(x+3)^2 = 1$$
$$\frac{9y^2}{1} - \frac{1(x+3)^2}{1} = \frac{1}{1}$$
$$\frac{y^2}{\frac{1}{9}} - \frac{(x+3)^2}{1} = 1$$

Center: (-3, 0)

Vertices $\left(-3, \frac{1}{3}\right)$ & $\left(-3, -\frac{1}{3}\right)$

Foci $\left(-3, \pm\frac{\sqrt{10}}{3}\right)$

Asymptotes: $y = \pm\frac{1}{3}(x+3)$

Hyperbolas can be used in so-called ‘trilateration’ or ‘positioning’ problems. The procedure outlined in the next example is the basis of the Long Range Aid to Navigation (LORAN) system, (outdated now due to GPS)

Ex. Jeff is stationed 10 miles due west of Carl in an otherwise empty forest in an attempt to locate an elusive Sasquatch. At the stroke of midnight, Jeff records a Sasquatch call 9 seconds earlier than Carl. Kai is also camping in the woods, he is 6 miles due north of Jeff and heard the Sasquatch call 18 seconds after Jeff did. If the speed of sound that night is 760 miles per hour, determine the location of the Sasquatch.

Relationship between Jeff and Carl:

$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\text{ hour}}{3600\text{ seconds}} \times 9\text{ seconds} = 1.9\text{ miles}$ is the constant *d, d = 2a*

a = 0.95

c = 5

$c^2 = a^2 + b^2$, $\therefore b^2 = 24.0975$

$$\frac{x^2}{0.9025} - \frac{y^2}{24.0975} = 1$$

Relationship between Jeff and Kai:

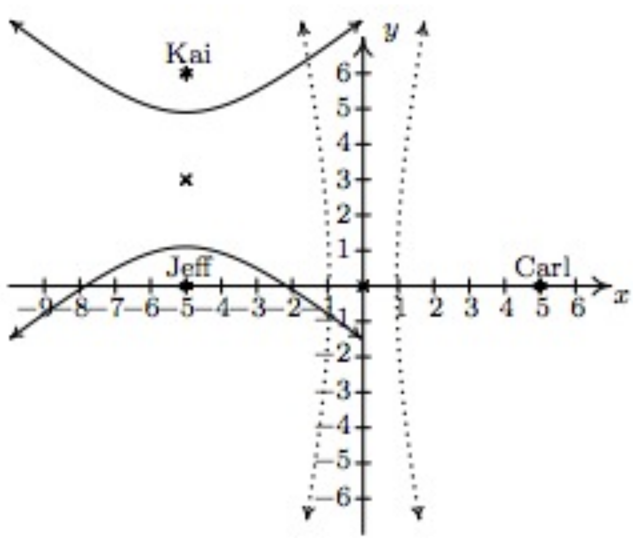
$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\text{ hour}}{3600\text{ seconds}} \times 18\text{ seconds} = 3.8\text{ miles}$ is the constant *d, d = 2a*

a = 1.9

c = 3

$\therefore b^2 = 5.391$

$$\frac{(y-3)^2}{3.61} - \frac{(x+5)^2}{5.39} = 1$$



Using a graphing utility we find Sasquatch located at $(-0.9629, -0.8113)$. Without a graphing utility each hyperbola would need to be written in the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ and use techniques for solving systems of non-linear equations, matrices.

Homework:

In Exercises 1-4, graph the hyperbola. Find the center, lines which contain the transverse and conjugate axis, the vertices, the foci, and the equations of the asymptotes.

1. $\frac{x^2}{25} - \frac{y^2}{16} = 1$

2. $\frac{y^2}{16} - \frac{x^2}{25} = 1$

3. $\frac{(y-1)^2}{16} - \frac{x^2}{16} = 1$

4. $\frac{(x-2)^2}{\frac{3}{2}} - \frac{(y-1)^2}{\frac{3}{4}} = 1$

In Exercises 5-7, put the equation in standard form. Identify the location of the foci.

5. $12x^2 - 4y^2 + 40y - 148 = 0$

6. $18y^2 - 7x^2 + 72y + 42x - 117 = 0$

7. $25y^2 - 16x^2 + 200y + 64x - 64 = 0$

In Exercises 8-13, find the standard form of the equation of the hyperbola.

8. Center (2, 6), Vertex (2, 2), Focus (2, 1)

9. Foci $(0, \pm 6)$, Vertices $(0, \pm 4)$

10. Foci $(0, \pm 4)$, length of conjugate axis 6

11. Vertices (2, 2), (12, 2); Endpoints of Conjugate Axis (7, 4), (7, 0)

12. Vertex (-10, 4), Asympotes $y = \pm\frac{1}{2}(x+5) + 4$

13. The notion of eccentricity for ellipses is the same for hyperbolas in that we can define the eccentricity *e* for a hyperbola as

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

a. Explain why $e > 1$ for any hyperbola.

b. Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and eccentricity $e = 2$.

c. Find the eccentricity of each of the hyperbolas in Exercises 1-4.

d. Describe how the shape of the hyperbola changes as the eccentricity increases.

14. 'Natural Draft' cooling towers are often shaped as **hyperboloids of revolution.** Each vertical cross section of these towers is a hyperbola. Suppose the tower is 450 feet wide at the base, 275 feet wide at the top, and 220 feet at its narrowest point (which occurs 330 feet above the ground.) Determine the height of the tower to the nearest foot.